

## ALMOST OPEN AND ALMOST HOMEOMORPHISMS

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ABSTRACT. This paper is devoted to the study of various notions of almost openness and almost homeomorphisms and the characterization of some of them in terms of the relative interior operator. Generally, openness (or quasi-openness) for a continuous map  $f$  is well known. We define openness (or quasi-openness) at a point  $x$  using the relative interior operator and characterize that a continuous map  $f$  is almost open, almost quasi-open, almost embedding and almost homeomorphisms.

### 1. Introduction and preliminaries

The main purpose of this paper is to extend the following Theorem A and Theorem B for almost continuity maps and homeomorphisms to open maps and almost homeomorphisms.

Theorem A [6]. Let  $E$  be a Baire Space,  $F$  be a second countable space and  $f$  be a mapping of  $E$  into  $F$ . Then the set of points of almost continuity of  $f$  is dense in  $E$ .

Theorem B [8]. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous. If  $g \circ f : X \rightarrow Z$  is a homeomorphism, then  $g$  one-to-one (or  $f$  onto) implies that  $f$  and  $g$  are homeomorphisms.

We prove here the following results.

Theorem A'. Let  $f : X_1 \rightarrow X_2$  be a continuous, closed and proper map. For every  $V_1 \in \mathcal{U}_{X_1}$  the set  $\text{Int}\{y \in X_2 : f \text{ is } V_1 \text{ open at each } x \in f^{-1}(y)\}$  is an open and dense subset of  $X_2$ . In particular, if  $X_1$  is metrizable and  $X_2$  is Baire, then  $\{y \in X_2 : f \text{ is } V_1 \text{ open at each } x \in f^{-1}(y)\}$  is a residual subset of  $X_2$ .

Theorem B'. Let  $f : X_1 \rightarrow X_2$  and  $g : X_2 \rightarrow X_3$  be continuous maps. Then

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- (1) Assume  $f$  is surjective.  $g \circ f$  is an almost homeomorphism if and only if both  $f$  and  $g$  are almost homeomorphisms.
- (2) Assume  $g$  is an almost homeomorphism. If  $g \circ f$  is almost open, then  $f$  is almost quasi-open.
- (3) Assume  $g$  is an almost homeomorphism. If  $g \circ f$  is almost quasi-open and  $g$  is closed, then  $f$  is almost quasi-open.

We now introduce notions and definitions necessary for our works. Throughout the present paper,  $X_1$  and  $X_2$  always mean topological spaces and by  $f : X_1 \rightarrow X_2$  we denote a map. Let  $A$  and  $B$  be subsets of  $X$ . The closure of  $A$  and the interior of  $A$  are denoted by  $\bar{A}$  and  $\text{Int}A$ , respectively. A subset  $B$  of  $A$  is dense in  $A$  means that  $\bar{A} \subset \bar{B}$ . A subset  $A$  of  $X$  is said to be pre-open if  $A \subset \text{Int}\bar{A}$  in [6]. This set is called by quasi-open in [2].

A relation  $F : X_1 \rightarrow X_2$  is considered a map from  $X_1$  to the power set of  $X_2$ , that is, each  $x \in X_1$  corresponds to a subset  $F(x)$  of  $X_2$ , or a subset of  $X_1 \times X_2$  so that  $y \in F(x)$  means  $(x, y) \in F$ .

For relations  $F : X_1 \rightarrow X_2$  and  $G : X_2 \rightarrow X_3$  we define the *inverse*  $F^{-1} : X_2 \rightarrow X_1$  and the *composition*  $G \circ F$  (simply  $GF$ ):  $X_1 \rightarrow X_3$  by

$$x \in F^{-1}(y) \iff y \in F(x), \text{ i.e., } F^{-1} = \{(y, x) \mid (x, y) \in F\}.$$

$$y \in (GF)(x) \iff z \in F(x) \text{ and } y \in G(z) \text{ for some } z \in X_2.$$

In other words,  $GF$  is the projection to  $X_1 \times X_3$  of the subset  $\{(x, z, y) \in X_1 \times X_2 \times X_3 \mid (x, z) \in F \text{ and } (z, y) \in G\}$ .

A relation  $F : X_1 \rightarrow X_2$  is called a *closed relation* if it is a closed subset of  $X_1 \times X_2$ . It is a *pointwise closed relation* if  $F(x)$  is a closed subset of  $X_2$  for every  $x \in X_1$ . Clearly, a closed relation is a pointwise closed relation.  $F$  is called a *compact relation* if  $F(x)$  is a compact subset of  $X_2$  for any  $x \in X_1$ .

Remark that in general, a compact relation need not be a closed relation. For example, let  $X = [0, 1]$  and  $F = \{([0, \frac{1}{2}) \cup (\frac{1}{2}, 1]) \times [\frac{1}{3}, \frac{2}{3}]\} \cup \{(\frac{1}{2}, \frac{1}{2})\}$ . Then  $F$  is a compact and pointwise closed relation but  $F$  is not a closed relation. We are concerned with subsets of a cartesian product  $X \times X$  of a set with itself. These subsets are relations on  $X$ . If  $U = U^{-1}$ , then  $U$  is called *symmetric*. The set of all pairs  $(x, x)$  for  $x$  in  $X$  is called the *identity relation*, or the *diagonal*, and is denoted by  $\Delta(X)$  or simply  $\Delta$ . For each subset  $A$  of  $X$  the set  $U(A)$  is defined to be  $\{y : (x, y) \in U \text{ for some } x \text{ in } A\}$ , and if  $x$  is a point of  $X$ , then  $U(x)$  is  $U(\{x\})$ . For each  $U$  and  $V$  and each  $A$  it is true that  $(U \circ V)(A) = U(V(A))$ . Finally, a simple definition will be needed.

DEFINITION 1.1. [4] A *uniformity* for a set  $X$  is a non-void family  $\mathcal{U}_X$  of subsets of  $X \times X$  such that

- (1) each member of  $\mathcal{U}_X$  contains the diagonal  $\Delta$ ;
- (2) If  $U \in \mathcal{U}_X$ , then  $U^{-1} \in \mathcal{U}_X$ ;
- (3) if  $U \in \mathcal{U}_X$ , then there exists  $V \in \mathcal{U}_X$  such that  $VV = V^2 \subset U$ ;
- (4) if  $U$  and  $V$  are members of  $\mathcal{U}_X$ , then  $U \cap V \in \mathcal{U}_X$ ; and
- (5) if  $U \in \mathcal{U}_X$  and  $U \subset V \subset X \times X$ , then  $V \in \mathcal{U}_X$ .

The pair  $(X, \mathcal{U}_X)$  (simply denote  $\mathcal{U}_X$ ) is a *uniform space*.

The metric antecedents of the conditions above are not hard to discern. The first is derived from the condition that  $d(x, x) = 0$  and the second derives from the condition that  $d(x, y) = d(y, x)$ . The third is a vestigial form of the triangle inequality - it says roughly that for  $r$ -spheres there are  $(r/2)$ - spheres. The fourth and fifth resemble axioms for the neighborhood system of a point and they will be used to derive the corresponding properties for a neighborhood system relative to a topology which will presently be defined.

There may be many different uniformities for a set  $X$ . The largest of these is the family of all those subsets of  $X \times X$  which contain  $\Delta$  and the smallest is the family whose only member is  $X \times X$ . If  $X$  is the set of real numbers the *usual uniformity* for  $X$  is the family  $\mathcal{U}_X$

## 2. Semi-continuous relations

In this section, we develop the fundamentals of the upper and lower semi-continuous relations.

Let  $f : X_1 \rightarrow X_2$  be a map. We define the *equivalence relation*:

$$E_f = (f \times f)^{-1}(1_{X_2}) = \{(x_1, x_2) : f(x_2) = f(x_1)\} = f^{-1} \circ f.$$

DEFINITION 2.1. Let  $F : X_1 \rightarrow X_2$  and  $H : X_2 \rightarrow X_2$  be relations. we define the relation  $F^*H$  on  $X_1$  by:

$$F^*H = \{(x_1, x_2) : F(x_2) \subset (H \circ F)(x_1)\}.$$

PROPOSITION 2.2. Let  $F : X_1 \rightarrow X_2$  be a relation. The following properties hold:

- (1) For relations  $H_1, H_2$  on  $X_2$  we have  
 $H_1 \subset H_2$  implies  $F^*H_1 \subset F^*H_2$  and  
 $F^*(H_1)F^*(H_2) \subset F^*(H_1H_2)$ .
- (2) For relations  $H$  on  $X_2$  and  $K$  on  $X_1$ :  
 $FK \subset HF$  if and only if  $K \subset F^*H$ .

For any relation  $H$  on  $X_2$ :

$$F(F^*H) \subset HF.$$

with equality when  $F$  is a surjective map.

(3)  $1_{X_2}^*H = H$ , and if  $G : X_0 \rightarrow X_1$  and  $F : X_1 \rightarrow X_2$  are relations, then:

$$G^*(F^*H) \subset (FG)^*H$$

with equality when  $G$  is a map.

(4) For a map  $f : X_1 \rightarrow X_2$  and relations  $F : X_0 \rightarrow X_1$  and  $G : X_0 \rightarrow X_2$ :

$$fF \subset G \iff F \subset f^{-1}G.$$

(5) For a map  $f : X_1 \rightarrow X_2$  and relation  $H$  on  $X_2$ :

$$f^*H = f^{-1}Hf = (f \times f)^{-1}(H) = \{(x_1, x_2) : f(x_2) \in H(f(x_1))\}.$$

So when  $f$  is a map:

$$\begin{aligned} f^*(H^{-1}) &= (f^*H)^{-1} \text{ and} \\ E_f(f^*H)E_f &= f^*H. \end{aligned}$$

*Proof.* (1) By the definition of  $F^*H$ ,  $F^*H_1 \subset F^*H_2$  for every  $H_1 \subset H_2$ . Let  $(x_1, x_3) \in F^*(H_1)F^*(H_2)$  be given. By the definition of composition, we can find  $x_2 \in X_2$  such that  $(x_1, x_2) \in F^*(H_2)$  and  $(x_2, x_3) \in F^*(H_1)$ . In other words,  $F(x_2) \subset H_2F(x_1)$  and  $F(x_3) \subset H_1F(x_2)$ . Hence  $F(x_3) \subset H_1F(x_2) \subset H_1H_2F(x_1)$ .

(2) Let  $A$  be a subset of  $X_1$  and  $x \in A$ . If  $F(A) \subset HF(x_1)$ , then  $F(x) \subset HF(x_1)$ . Hence  $(x_1, x) \in F^*H$ , i.e.,  $x \in (F^*H)(x_1)$ . Conversely, we assume that  $A \subset (F^*H)(x_1)$ . If  $y \in F(A)$ , then there exists  $x \in A \subset (F^*H)(x_1)$  such that  $y \in F(x) \subset HF(x_1)$ . Hence we obtain the following property.

$$F(A) \subset HF(x_1) \iff A \subset (F^*H)(x_1).$$

Replace  $A$  by  $K(x_1)$  and we get:

$$(FK)(x_1) \subset (HF)(x_1) \iff K(x_1) \subset (F^*H)(x_1).$$

Hence  $FK \subset HF \iff K \subset F^*H$ .

In particular, with  $K = F^*H$  we have  $FK \subset HF$  if and only if  $K \subset F^*H$ . For the surjective map case, we first consider (5).

(5) It is clear that  $(x_1, x_2) \in f^*H$  if and only if  $f(x_2) \in H(f(x_1))$ . This says  $f^*H = (f \times f)^{-1}(H)$  and  $f^*H = f^{-1}Hf$ ,  $f^*H = f^{-1}Hf = (f \times f)^{-1}(H) = \{(x_1, x_2) : f(x_2) \in H(f(x_1))\}$ . From this  $f^*(H^{-1}) = (f^*H)^{-1}$  is obvious.

When  $f$  is a surjective map,  $ff^*H = ff^{-1}Hf = Hf$  by  $ff^{-1} = 1_{X_2}$ .

(3)  $1_{X_2}^*H = H$  is clear. By (2),  $FG(G^*(F^*H)) \subset F(F^*H)G \subset HFG$ . The  $G^*F^*H \subset (FG)^*H$  by (2). If  $G = g$  is a map,  $(Fg)((Fg)^*H)g^{-1} \subset (H(Fg))g^{-1} \subset HF$  and  $g((Fg)^*H)g^{-1} \subset F^*H$ . By (5),  $(Fg)^*H \subset g^{-1}g((Fg)^*H)g^{-1}g \subset g^{-1}(F^*H)g = g^*(F^*H)$ . Alternatively, observe that  $(x_1, x_2) \in (Fg)^*H$  if and only if  $F(g(x_2)) \subset HF(g(x_1))$  and so if and only if  $(g(x_1), g(x_2)) \in F^*H$ .

(4) Recall that for subsets  $A$  of  $X_1$  and  $B$  of  $X_2$   $f(A) \subset B \iff A \subset f^{-1}(B)$ . Put  $A = F(x_0)$  and  $B = G(x_0)$  for given  $x_0 \in X_0$ . Then  $fF \subset G \iff F \subset f^{-1}G$ .

(5) For a map  $f$ ,  $E_f = (f \times f)^{-1}(1_{X_2}) = \{(x_1, x_2) : f(x_1) = f(x_2)\} = f^{-1}f = f^*1_{X_2}$ . Since  $f$  is a map,  $f = f1_{X_1} \subset ff^{-1}f \subset 1_{X_2}f = f$  and  $f^{-1} = 1_{X_1}f^{-1} \subset f^{-1}ff^{-1} \subset f^{-1}1_{X_2} = f^{-1}$ . Hence  $fE_f = ff^{-1}f = f$  and  $E_ff^{-1} = f^{-1}ff^{-1} = f^{-1}$ .

Finally, by (1),  $E_f(f^*H)E_f = f^*1_{X_2}(f^*H)f^*1_{X_2} \subset f^*(1_{X_2}H)f^*1_{X_2} = (f^*H)(f^*1_{X_2}) \subset f^*(H1_{X_2}) = f^*(H)$ . Since  $1_{X_1} \subset f^{-1}f$  and  $E_f = f^{-1}f = f^*1_{X_2}$ ,  $f^*H = 1_{X_1}(f^*H)1_{X_1} \subset f^{-1}f(f^*H)f^{-1}f = E_f(f^*H)E_f$  and  $f^*(H^{-1}) = f^{-1}H^{-1}f = (f^{-1}Hf)^{-1} = (f^*H)^{-1}$ .  $\square$

DEFINITION 2.3. For a relation  $F : X_1 \rightarrow X_2$ ,  $V_2 \in \mathcal{U}_{X_2}$  and  $x \in X_1$  we call  $F$  is  $V_2$  upper semicontinuous at  $x$ , when  $x \in \text{Int}((F^*V_2)(x))$ ;  $V_2$  lower semicontinuous at  $x$ , when  $x \in \text{Int}((F^*V_2)^{-1}(x))$ ;  $V_2$  continuous at  $x$ , when  $(x, x) \in \text{Int}F^*V_2$  in  $X_1 \times X_1$ .  $F$  is upper semicontinuous / lower semicontinuous / continuous at  $x$  if it is  $V_2$  upper semicontinuous /  $V_2$  lower semicontinuous /  $V_2$  continuous at  $x$  for all  $V_2$  in  $\mathcal{U}_{X_2}$ , respectively.  $F$  is upper semicontinuous / lower semicontinuous / continuous if it satisfies the corresponding condition at every  $x$  in  $X_1$ .  $F$  is uniformly continuous provided that for every  $V_2 \in \mathcal{U}_{X_2}$  there exists  $V_1 \in \mathcal{U}_{X_1}$  such that  $FV_1 \subset V_2F$ .

The following examples show that  $F$  is upper semicontinuous but is not lower semicontinuous at  $x$ ,  $G$  is lower semicontinuous but is not upper semicontinuous at  $x$ , and  $H$  is continuous at  $x$ .

EXAMPLE 2.4. Define a relation  $F$  on  $\mathbb{R}$  by  $F = (-\infty, 0) \times \{1\} \cup [0, \infty) \times [\frac{1}{2}, \frac{3}{2}]$  and let  $\epsilon \in (0, \frac{1}{2})$  be given. By definition of  $F^*V_\epsilon$ , if  $(x, y) \in F^*V_\epsilon$ , then  $F(y) \subset V_\epsilon(F(x)) = B(F(x), \epsilon)$ , where  $V_\epsilon = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : d(x_1, x_2) < \epsilon\}$ . If  $x \in (-\infty, 0)$ , then  $F(y) \subset B(F(x), \epsilon) = B(\{1\}, \epsilon) = (1 - \epsilon, 1 + \epsilon) \subset (\frac{1}{2}, \frac{3}{2})$ . Hence  $y \in (-\infty, 0)$ . If  $x \in [0, \infty)$ , then  $F(y) \subset B(F(x), \epsilon) = B([\frac{1}{2}, \frac{3}{2}], \epsilon) = (\frac{1}{2} - \epsilon, \frac{3}{2} + \epsilon)$ . Hence  $y \in (-\infty, \infty)$ . This means that  $F^*V_\epsilon \subset (-\infty, 0) \times (-\infty, 0) \cup [0, \infty) \times (-\infty, \infty)$ . Let  $(x, y) \in (-\infty, 0) \times (-\infty, 0) \cup [0, \infty) \times (-\infty, \infty)$  be given. If  $x < 0$  and  $y < 0$ , then  $F(y) = \{1\} \subset (1 - \epsilon, 1 + \epsilon) = B(\{1\}, \epsilon) = B(F(x), \epsilon) = V_\epsilon(F(x))$ . Hence  $(x, y) \in F^*V_\epsilon$ . If  $x \geq 0$  and  $y \in \mathbb{R}$ ,  $F(y) \subset [\frac{1}{2}, \frac{3}{2}] \subset (\frac{1}{2} - \epsilon, \frac{3}{2} + \epsilon) = B([\frac{1}{2}, \frac{3}{2}], \epsilon) = B(F(x), \epsilon) = V_\epsilon(F(x))$ . Hence  $(x, y) \in F^*V_\epsilon$ . Finally, we know that  $F^*V_\epsilon = (-\infty, 0) \times (-\infty, 0) \cup [0, \infty) \times (-\infty, \infty)$ . Since  $F^*V_\epsilon(0) = (-\infty, \infty)$ ,  $0 \in (-\infty, \infty) = \text{Int}F^*V_\epsilon(0)$ . Therefore  $F$  is  $V_\epsilon$  upper semicontinuous at 0. Since  $(F^*V_\epsilon)^{-1} = (-\infty, 0) \times$

$(-\infty, 0) \cup (-\infty, \infty) \times [0, \infty)$ ,  $(F^*V_\epsilon)^{-1}(0) = [0, \infty)$ . It means that  $0 \notin (0, \infty) = \text{Int}(F^*V_\epsilon)^{-1}(0)$ . Consequently,  $F$  is not  $V_\epsilon$  lower semicontinuous at 0.

EXAMPLE 2.5. Define a relation  $G$  on  $\mathbb{R}$  by  $G = (-\infty, 0] \times \{1\} \cup (0, \infty) \times [\frac{1}{2}, \frac{3}{2}]$  and let  $\epsilon \in (0, \frac{1}{2})$  be given. Since  $G^*V_\epsilon = (-\infty, 0] \times (-\infty, 0] \cup (0, \infty) \times (-\infty, \infty)$ ,  $G^*V_\epsilon(0) = (-\infty, 0]$ . We know that  $G$  is not  $V_\epsilon$  upper semicontinuous at 0 because  $0 \notin (-\infty, 0) = \text{Int}G^*V_\epsilon(0)$ . Since  $(G^*V_\epsilon)^{-1} = (-\infty, 0] \times (-\infty, 0] \cup (-\infty, \infty) \times (0, \infty)$ ,  $(G^*V_\epsilon)^{-1}(0) = (-\infty, \infty)$ . It means that  $G$  is  $V_\epsilon$  lower semicontinuous at 0 because  $0 \in (-\infty, \infty) = \text{Int}(G^*V_\epsilon)^{-1}(0)$ .

EXAMPLE 2.6. Define a relation  $H$  on  $\mathbb{R}$  by  $H = (-\infty, \infty) \times [\frac{1}{2}, \frac{3}{2}]$  and let  $\epsilon > 0$  be given. Since  $H^*V_\epsilon = (-\infty, \infty) \times (-\infty, \infty)$ ,  $H$  is  $V_\epsilon$  continuous at 0 because  $(0, 0) \in (-\infty, \infty) \times (-\infty, \infty) = \text{Int}H^*V_\epsilon$ .

LEMMA 2.7. Let  $F : X_1 \rightarrow X_2$  be a relation and  $V_2 \in \mathcal{U}_{X_2}$ .

(1) Let  $\tilde{V}_2 \subset V_2$ . If  $F$  is  $\tilde{V}_2$  upper semicontinuous at  $x / \tilde{V}_2$  lower semicontinuous at  $x / \tilde{V}_2$  continuous at  $x$ , then  $F$  satisfies the corresponding property for  $V_2$ .

(2) If  $F$  is  $V_2$  continuous at  $x$ , then  $F$  is  $V_2$  upper semicontinuous at  $x$  and  $V_2$  lower semicontinuous at  $x$ . If  $\tilde{V}_2 \in \mathcal{U}_{X_2}$  with  $\tilde{V}_2^2 \subset V_2$  and  $F$  is  $\tilde{V}_2$  upper semicontinuous at  $x$  and  $\tilde{V}_2$  lower semicontinuous at  $x$ , then  $F$  is  $V_2$  continuous at  $x$ .

(3)  $F$  is continuous at  $x$  if and only if  $F$  is upper semicontinuous at  $x$  and lower semicontinuous at  $x$ . If  $F$  is uniformly continuous, then  $F$  is continuous. If  $F$  is continuous and  $X_1$  is compact, then  $F$  is uniformly continuous.

*Proof.* (1) This follows from Proposition 2.2 (1).

(2) If  $F$  be  $V_2$  continuous at  $x$ ,  $(x, x) \in \text{Int}F^*V_2$ . This means that  $x \in \text{Int}(F^*V_2)(x)$  and  $x \in \text{Int}(F^*V_2)^{-1}(x)$ . Therefore,  $F$  is  $V_2$  upper semicontinuous at  $x$  and  $V_2$  lower semicontinuous at  $x$ . Let  $\tilde{V}_2 \in \mathcal{U}_{X_2}$  with  $\tilde{V}_2^2 \subset V_2$ ,  $F$  be  $\tilde{V}_2$  upper semicontinuous at  $x$  and  $\tilde{V}_2$  lower semicontinuous at  $x$ . Since  $\tilde{V}_2 \subset V_2$ ,  $x \in \text{Int}((F^*V_2)(x))$  and  $x \in \text{Int}((F^*V_2)^{-1}(x))$ . It means that  $F$  is  $V_2$  continuous at  $x$ .

(3) By Definition 2.3, the proof of (3) is obvious.  $\square$

THEOREM 2.8. Let  $F : X_1 \rightarrow X_2$  be a pointwise closed relation. Assume that  $F$  is upper semicontinuous and  $F(x)$  is compact for every  $x \in X_1$ . For every  $V_2 \in \mathcal{U}_{X_2}$  the set of  $V_2$  continuity points,  $\{x : (x, x) \in \text{Int}F^*V_2\}$ , is open and dense in  $X_1$ .

*Proof.* The set of  $V_2$  continuity points,  $\{x : (x, x) \in \text{Int}F^*V_2\}$ , is always open. By Lemma 2.7 (2), it suffices to prove that the set of  $V_2$  upper semicontinuous points and the set of  $V_2$  lower semicontinuous points are each dense.

Fix  $V_2 \in \mathcal{U}_{X_2}$ ,  $x_1 \in X_1$ , and  $O_1$  an open set containing  $x_1$ . We claim that  $O_1 \cap \{x : (x, x) \in \text{Int}F^*V_2\} \neq \emptyset$ .

We produce  $x_0 \in O_1$  at which  $F$  is  $V_2$  lower semicontinuous. Choose  $\tilde{V}, W \in \mathcal{U}_{X_2}$ , symmetric, with  $\tilde{V}$  closed and such that  $\tilde{V}^2 \subset V_2, W^2 \subset \tilde{V}$  and  $W^2 \subset \tilde{V}$ . Because  $F$  is upper semicontinuous, there is an open set  $\tilde{O} \subset O_1$  such that  $x \in \tilde{O}$  implies  $F(x) \subset W(F(x_1))$ , i.e.  $(x_1, x) \in F^*W$ . Because  $F(x_1)$  is compact, it has a finite subset  $R$  such that  $F(x_1) \subset W(R)$ . Define for  $x \in \tilde{O}$ :

$$R(x) = R \cap \tilde{V}(F(x)).$$

Clearly, for  $x \in \tilde{O}$ :

$$(2.1) \quad R(x) \subset \tilde{V}(F(x)) \quad \text{and} \quad (R - R(x)) \cap \tilde{V}(R(x)) = \emptyset.$$

If there exists  $y \in F(x) \cap \tilde{V}(R - R(x))$ , then  $y \in F(x)$  and  $y \in \tilde{V}(F(x))$ . This is a contradiction for  $(R - R(x)) \cap \tilde{V}(R(x)) = \emptyset$ . Hence we get:

$$(2.2) \quad \begin{aligned} F(x) \cap \tilde{V}(R - R(x)) &= \emptyset \quad \text{and} \quad F(x) \subset \tilde{V}(R(x)); \\ F(x) \cap \tilde{V}(R - R(x)) &= \emptyset \quad \text{and} \quad F(x) \subset \tilde{V}(R(x)). \end{aligned}$$

Since  $F$  is upper semicontinuous and  $F(x_1) \subset W(R)$ , we obtain the following property.

$$F(x) \subset W(F(x_1)) \subset W^2(R) \subset \tilde{V}(R(x)) \cup \tilde{V}(R - R(x)).$$

Now choose  $x_0 \in \tilde{O}$  so that  $R(x_0)$  is minimal in the family  $\{R(x) : x \in \tilde{O}\}$  of subsets of the finite set  $R$ .

Since  $F(x_0)$  is compact and  $\tilde{V}(R - R(x_0))$  is the finite union of closed sets, and thus is closed, there exists  $W_0 \in \mathcal{U}_{X_2}$  such that

$$W_0(F(x_0)) \cap \tilde{V}(R - R(x_0)) = \emptyset.$$

Since  $F$  is upper semicontinuous:

$$O_0 \equiv \{x \in \tilde{O} : (x_0, x) \in F^*W_0\}$$

is a neighborhood of  $x_0$ . For  $x \in O_0$ ,  $W_0(F(x_0)) \cap \tilde{V}(R - R(x_0)) = \emptyset$  implies  $R(x) \cap (R - R(x_0)) = \emptyset$ , so  $R(x) \subset R(x_0)$ . By the minimality of  $R(x_0)$ , we have  $R(x) = R(x_0)$  for all  $x \in O_0$ . Therefore we obtain that

$$F(x_0) \subset \tilde{V}(R(x_0)) = \tilde{V}(R(x)) \subset \tilde{V}^2(F(x)) \subset V_2(F(x))$$

for all  $x \in O_0$ .

This means  $O_0 \times \{x_0\} \subset F^*V_2$ , so  $F$  is  $V_2$  lower semicontinuous at  $x_0$ , i.e.  $x_0 \in O_1 \cap \{x : (x, x) \in \text{Int}F^*V_2\}$ . Hence the set of  $V_2$  upper

semicontinuous points is dense. Similarly, we can derive the fact that the set of  $V_2$  lower semicontinuous points is dense.  $\square$

**THEOREM 2.9.** *Let  $F : X_1 \rightarrow X_2$  be a pointwise closed relation. Assume that  $F$  is lower semicontinuous and  $X_2$  is compact. For every  $V_2 \in \mathcal{U}_{X_2}$  the set of  $V_2$  continuity points,  $\{x : (x, x) \in \text{Int}F^*V_2\}$ , is open and dense in  $X_1$ .*

*Proof.* If  $x_1 \in X_1$  and  $O_1$  is an open set containing  $x_1$ , then we can choose  $\tilde{V}$ ,  $W \in \mathcal{U}_{X_2}$  symmetric with  $\tilde{V}$  open,  $\tilde{V}^2 \subset V_2$ , and  $W^2 \subset \tilde{V}$ . Choose  $R$  a finite subset of  $X_2$  such that  $W(R) = X_2$ , and define  $R(x)$  for all  $x \in X_2$  according to  $R(x) = R \cap \tilde{V}(F(x))$ . Then for all  $x \in X_2$ , (2-1) and (2-2) hold as before.

Choose  $x_0 \in O_1$  so that  $R(x_0)$  is maximal in the family  $\{R(x) : x \in O_1\}$ . Since  $\tilde{V}$  is open and  $F(x_0)$  is compact, there exists  $W_0 \in \mathcal{U}_{X_2}$  symmetric such that for each  $y \in R(x_0)$  there exists  $z_y \in F(x_0)$  such that  $W_0(z_y) \subset \tilde{V}(y)$ . Because  $F$  is lower semicontinuous:

$$O_0 \equiv \{x \in O_1 : F(x_0) \subset W_0(F(x))\}$$

is a neighborhood of  $x_0$ . For each  $x \in O_0$  and  $y \in R(x_0)$ ,  $\tilde{V}(y)$  contains  $W_0(z_y)$ , which meets  $F(x)$ . Thus  $y \in \tilde{V}(F(x))$ . Consequently,  $R(x_0) \subset R(x)$ , so by the maximality of  $R(x_0)$ , we have  $R(x_0) = R(x)$  for  $x \in O_0$ . Hence for all  $x \in O_0$ :

$$F(x) \subset \tilde{V}(R(x)) = \tilde{V}(R(x_0)) \subset \tilde{V}^2(F(x_0)) \subset V_2(F(x_0)).$$

This says  $\{x_0\} \times O_0 \subset F^*V_2$ , so  $F$  is upper semicontinuous at  $x_0$ .  $\square$

**LEMMA 2.10.**  *$X$  is a Baire space if and only if given any countable collection  $\{U_n\}$  of open sets in  $X$ , each of which is dense in  $X$ , their intersection  $\cap U_n$  is also dense in  $X$ .*

*Proof.* See [9] Lemma 7.1.  $\square$

**LEMMA 2.11.** *Let  $f : X_1 \rightarrow X_2$  be a continuous map.*

(1) *Let  $f$  be a closed map. Assume  $A$  is an  $E_f$  invariant set, i.e.,  $A = E_f(A)$ . If  $B$  is closed and  $B \subset A$ , then  $B_1 = E_f(B)$  is an  $E_f$  invariant closed subset of  $X_1$  satisfying  $B \subset B_1 \subset A$ . If  $O$  is open and  $A \subset O$ , then  $O_1 = \{x : E_f(x) \subset O\}$  is an  $E_f$  invariant open subset of  $X_1$  satisfying  $A \subset O_1 \subset O$ . The relations  $E_f$  on  $X_1$  and  $f^{-1} : X_2 \rightarrow X_1$  are upper semicontinuous.*

(2) *If  $f$  is a proper map (i.e., point inverses are compact) and  $f^{-1}$  is upper semicontinuous, then  $f$  is a closed map.*

*Proof.* (1) If  $B$  is closed, then  $B_1 = E_f(B) = f^{-1}f(B)$  is closed when  $f$  is a continuous, closed map.  $B \subset A = E_f(A)$  implies  $B \subset$



$E_f(B) \subset E_f(A) = A$ . Since  $O$  is open,  $O_1 = \{x : E_f(x) \subset O\} = X_1 - E_f(X - O)$  is open. If  $A$  is invariant, then  $A_1 = \{x : E_f(x) \subset A\} = A$ . Therefore  $A \subset O$  implies  $A = A_1 \subset O_1 \subset O$ . Similarly,  $\{y \in X_2 : f^{-1}(y) \subset O\} = X_2 - f(X_1 - O)$  is open in  $X_2$ . In particular, for any  $V_1 \in \mathcal{U}_{X_1}$  and  $y \in X_2$ ,  $y \in \{y_1 \in X_2 : f^{-1}(y_1) \subset \text{Int}V_1(f^{-1}(y))\} \subset ((f^{-1})^*V_1)(y)$ . Then  $y \in \text{Int}((f^{-1})^*V_1)(y)$  for any  $V_1 \in \mathcal{U}_{X_1}$ . Thus  $f^{-1}$  is upper semicontinuous at  $y$ . By Proposition 2.2 (3), since  $f$  is a map,  $E_f^*V_1 = f^*(f^{-1})^*V_1 = f^{-1}((f^{-1})^*V_1)f$ . Hence for any  $x \in X_1$ ,  $(E_f^*V_1)(x) = f^{-1}((f^{-1})^*V_1(f(x)))$ . Then  $f(x) \in \text{Int}(f^{-1})^*V_1(f(x))$  implies  $x \in \text{Int}(E_f^*V_1)(x)$ , so  $E_f$  is upper semicontinuous at  $x$ .

(2) If  $A$  is closed in  $X_1$ ,  $y \notin f(A)$  and  $f^{-1}(y)$  is compact, then there exists  $V_1 \in \mathcal{U}_{X_1}$  such that  $V_1(f^{-1}(y)) \cap A = \emptyset$ . If  $f^{-1}$  is upper semicontinuous at  $y$ , then there exists  $O_2$  open with  $y \in O_2$  and  $f^{-1}(O_2) \subset V_1(f^{-1}(y))$ . Then  $O_2 \cap f(A) = \emptyset$  and  $y \notin f(A)$ . Hence  $f$  is a closed map. □

### 3. Genericity and almost homeomorphisms

In this section, we define openness (or quasi-openness) at a point  $x$  using the relative interior operator and characterize that a continuous map  $f$  is almost open, almost quasi-open, almost embedding and almost homeomorphsim.

For a continuous map  $f : X_1 \rightarrow X_2$  and let  $A \subset X_1$ , we define the *relative interior operator*:

$$\text{Int}_f A = (\text{Int}A) \cap f^{-1}(\text{Int}f(A)).$$

Clearly,  $\text{Int}_f A$  is an open subset of  $\text{Int}A$ . For  $x \in X_1$ ,  $x \in \text{Int}_f A$  if and only if  $x \in \text{Int}A$  and  $f(x) \in \text{Int}f(A)$ .

DEFINITION 3.1. Let  $f : X_1 \rightarrow X_2$  be a continuous map. For  $V_1 \in \mathcal{U}_{X_1}$  and  $x \in X_1$ , we call  $f$  is  $V_1$  open at  $x$  if  $x \in \text{Int}_f V_1(x)$ .  $f$  is open at  $x$  provided it is  $V_1$  open at  $x$  for all  $V_1 \in \mathcal{U}_{X_1}$  and  $f$  is open if  $f$  is open at  $x$  for every  $x \in X_1$ .

The equivalent conditions of  $V_1$  openness are the following.

LEMMA 3.2. Let  $f : X_1 \rightarrow X_2$  be a continuous map. Then the following statements are equivalent:

- (1)  $f$  is  $V_1$  open at  $x$ .
- (2) If  $U$  is a neighborhood of  $x$  in  $X_1$ , then  $f(U)$  is a neighborhood of  $f(x)$  in  $X_2$ .
- (3) For any subset  $A$  of  $X_1$ ,  $x \in \text{Int}A$  implies  $x \in \text{Int}_f A$ .

*Proof.* (1) $\Rightarrow$ (2). If  $f$  is  $V_1$  open at  $x$  for all  $V_1 \in \mathcal{U}_{X_1}$ , then  $x \in \text{Int}_f V_1(x)$ . Let  $U$  be a neighborhood of  $x$  in  $X_1$ . By the definition of the relative interior operator,  $x \in f^{-1}(\text{Int}f(V_1(x)))$ . It means that  $f(x) \in f(f^{-1}(\text{Int}f(V_1(x)))) \subset \text{Int}f(V_1(x)) \subset f(U)$ .

(2) $\Rightarrow$ (3). Let  $A$  be a subset of  $X_1$  and  $x \in \text{Int}A$ . By the hypothesis,  $f(\text{Int}A)$  is a neighborhood of  $f(x)$ . It means that  $x \in f^{-1}(\text{Int}f(A))$ . Thus,  $x \in \text{Int}_f A$ .

(3) $\Rightarrow$ (1). Since  $x \in V_1(x)$ ,  $x \in \text{Int}V_1(x)$ . By (3),  $x \in \text{Int}_f V_1(x)$ . It means that  $f$  is  $V_1$  open at  $x$ . □

**LEMMA 3.3.** *Let  $f : X_1 \rightarrow X_2$  be a continuous map and let  $y \in X_2$  and  $V_1$  be a symmetric element of  $\mathcal{U}_{X_1}$ . If  $f^{-1}$  is  $V_1$  lower semicontinuous at  $y$ , then  $f$  is  $V_1$  open at  $x$  and  $E_f$  is  $V_1$  lower semicontinuous at  $x$  for all  $x \in f^{-1}(y)$ . If  $f^{-1}(y)$  is compact and  $f$  is  $V_1$  open at every  $x \in f^{-1}(y)$ , then  $f^{-1}$  is  $V_1^2$  lower semicontinuous at  $y$ . In particular, if  $f^{-1}(y) = \emptyset$ , then  $f^{-1}$  is lower semicontinuous at  $y$ .*

*Proof.* If  $f^{-1}$  is  $V_1$  lower semicontinuous at  $y$ , then there exists  $V_2 \in \mathcal{U}_{X_2}$  such that  $V_2(y) \subset ((f^{-1})^*V_1)^{-1}(y)$ . This is,  $y_1 \in V_2(y)$  implies  $f^{-1}(y) \subset V_1(f^{-1}(y_1))$ , so  $V_2(y) \subset f(V_1(x))$  for all  $x \in f^{-1}(y)$ . For  $x_1 \in f^{-1}(V_2(y))$  setting  $y_1 = f(x_1)$  shows  $E_f(x) = f^{-1}(y) \subset V_1(E_f(x_1))$ . Thus  $E_f$  is  $V_1$  lower semicontinuous at  $x_1$ .

If  $f^{-1}(y)$  is compact, then there exists  $\{x_1, \dots, x_n\}$  in  $f^{-1}(y) \subset \cup_{i=1}^n V_1(x_i)$ . If  $f$  is  $V_1$  open at each  $x_i$ , then there exists  $V_2 \in \mathcal{U}_{X_2}$  such that  $V_2(y) \subset \cap_{i=1}^n f(V_1(x_i))$ . If  $y_1 \in V_2(y)$ ,  $f^{-1}(y_1) \cap V_1(x_i) \neq \emptyset$ , so  $f^{-1}(y_1) \cap V_1^2(x) \neq \emptyset$  for every  $x \in f^{-1}(y)$ . That is  $V_2(y) \subset ((f^{-1})^*(V_1^2))^{-1}(y)$  and  $f^{-1}$  is  $V_1^2$  lower semicontinuous at  $y$ . □

**PROPOSITION 3.4.** *Let  $f : X_1 \rightarrow X_2$  be a continuous map. Then the following statements are equivalent:*

- (1)  $f$  is open.
- (2) If  $U$  is open in  $X_1$ , then  $f(U)$  is open in  $X_2$ .
- (3) For all  $A \subset X_1$ ,  $\text{Int}A = \text{Int}_f A$ .

*Proof.* This is obvious from Lemma 3.2. □

**DEFINITION 3.5.** Let  $f : X_1 \rightarrow X_2$  be a continuous map. We call  $f$  is *almost open* provided that for all  $A \subset X_1$ ,  $\text{Int}A \neq \emptyset$  implies  $\text{Int}f(A) \neq \emptyset$ .

The equivalent conditions of almost openness are the following.

**THEOREM 3.6.** *Let  $f : X_1 \rightarrow X_2$  be a continuous map. Then the following statements are equivalent:*

- (1)  $f$  is almost open.

- (2)  $D$  dense in  $X_2$  implies  $f^{-1}(D)$  is dense in  $X_1$ .
- (3) If  $U$  is open in  $X_1$ , then  $\text{Int}_f U = U \cap f^{-1}(\text{Int} f(U))$  is dense in  $U$ .
- (4) For every  $V_1 \in \mathcal{U}_{X_1}$ ,  $\{x : f \text{ is } V_1 \text{ open at } x\}$  is dense in  $X_1$ .
- (5) For every  $V_1 \in \mathcal{U}_{X_1}$ ,  $\{x : (x, f(x)) \in \text{Int}(f \circ V_1)\}$  is open and dense in  $X_1$ .

*Proof.* (1) $\Rightarrow$ (2). Suppose that  $f^{-1}(D)$  is not dense in  $X_1$ . Then  $U = X_1 - f^{-1}(D)$  is a nonempty open set. By (1),  $\text{Int} f(U) \neq \emptyset$ . Since  $f(U) \cap D = \emptyset$ ,  $\text{Int} f(U) \cap D = \emptyset$ . This is a contradiction.

(2) $\Rightarrow$ (3). For any  $A \subset X_1$ ,  $D = (\text{Int} f(A)) \cup (X_2 - f(A))$  is dense in  $X_2$ . By (2),  $f^{-1}(A)$  is dense in  $X_1$ . If  $U$  is an open subset of  $X_1$ , then  $\text{Int}_f U = U \cap f^{-1}(D)$  is dense in  $U$ .

(3) $\Rightarrow$ (4). Let  $V_1 \in \mathcal{U}_{X_1}$  and  $x \in X_1$ . Also, let  $V_2$  be an open, symmetric element of  $\mathcal{U}_{X_1}$  such that  $V_2^2 \subset V_1$ . By (3),  $V_2(x) \cap f^{-1}(\text{Int} f(V_2(x)))$  is open and dense in  $V_2(x)$ . Choose  $x_1 \in V_2(x) \cap f^{-1}(\text{Int} f(V_2(x)))$ . Then  $f(x_1) \in f(f^{-1}(\text{Int} f(V_2(x)))) \subset \text{Int} f(V_2(x))$ . Since  $V_2$  can be chosen arbitrarily small,  $f$  is  $V_1$  open at points of a dense set.

(4) $\Rightarrow$ (1). For all  $A \subset X_1$ , let  $\text{Int} A \neq \emptyset$ . If  $V_1^2(x) \subset \text{Int} A$  let  $x_1 \in V_1(x)$  at which  $f$  is  $V_1$  open. Since  $f(V_1(x_1)) \subset f(V_1^2(x)) \subset f(A)$ ,  $f(V_1(x_1))$  is a neighborhood of  $f(x_1)$ .

(4)  $\iff$  (5). For  $V_1$  symmetric in  $\mathcal{U}_{X_1}$ ,  $\{x : (x, f(x)) \in \text{Int}(f \circ V_1)\} \subset \{x : f \text{ is } V_1 \text{ open at } x\} \subset \{x : (x, f(x)) \in \text{Int}(f \circ V_1^2)\}$ .

Observe that  $U_1 \times U_2 \subset f \circ V_1$  if and only if  $U_2 \subset f(V_1(x_1))$  for all  $x_1 \in U_1$ . The first inclusion is clear, and the second follows from  $f(V_1(x)) \subset f(V_1^2(x_1))$  for all  $x \in V_1(x)$ . Together these inclusions yield the equivalence.  $\square$

**THEOREM 3.7.** *Let  $f : X_1 \rightarrow X_2$  and  $g : X_2 \rightarrow X_3$  be continuous maps. If both  $f$  and  $g$  are almost open, then  $g \circ f$  is almost open. If  $g \circ f$  is almost open and  $f$  is surjective, then  $g$  is almost open.*

*Proof.* Let  $f$  and  $g$  be almost open and let  $\text{Int} A$  be a nonempty subset of  $X_1$ . Since  $f$  and  $g$  are almost open,  $\text{Int} f(A)$  and  $\text{Int} g(f(A))$  are nonempty subsets of  $X_2$  and  $X_3$ , respectively. Let  $g \circ f$  be almost open,  $f$  be surjective and  $\text{Int} A_2$  be a nonempty subset of  $X_2$ . Since  $f$  is continuous and surjective,  $f^{-1}(\text{Int} A_2)$  is a nonempty open subset of  $X_1$ .  $\text{Int}((g \circ f)(f^{-1}(\text{Int} A_2))) = \text{Int} g(\text{Int} A_2)$  is a nonempty subset of  $X_3$  because  $g \circ f$  is almost open. Therefore  $g$  is almost open.  $\square$

**THEOREM 3.8.** *Let  $f : X_1 \rightarrow X_2$  be a continuous, closed and proper map. For every  $V_1 \in \mathcal{U}_{X_1}$ , the set:*

$$\text{Int}\{y \in X_2 : f \text{ is } V_1 \text{ open at each } x \in f^{-1}(y)\}$$

is an open and dense subset of  $X_2$ . In particular, if  $X_1$  is metrizable and  $X_2$  is Baire, then

$$\{y \in X_2 : f \text{ is } V_1 \text{ open at each } x \in f^{-1}(y)\}$$

is a residual subset of  $X_2$ .

*Proof.* By Lemma 2.11 (1),  $f^{-1}$  is an upper semicontinuous relation. By Theorem 2.8,  $\{y : (y, y) \in \text{Int}(f^{-1})^*V_1\}$  is open and dense in  $X_2$  and it is contained in the set of points at which  $f^{-1}$  is a lower semicontinuous relation. By Lemma 3.3,  $f$  is  $V_1$  open at every point  $x \in X_1$  such that  $y = f(x)$  lies in this set. As usual when  $X_1$  is metrizable, we can intersect over a countable base for  $\mathcal{U}_{X_1}$  to obtain a residual subset of  $X_2$ .  $\square$

Generally  $\{x : f \text{ is almost open at } x\}$  is not dense when  $f$  is a continuous, closed and proper map.

EXAMPLE 3.9. Define  $f : [0, 1] \rightarrow [0, 2]$  by  $f(x) = 1$ . Then  $f$  is a continuous, closed and proper map. But  $\{x : f \text{ is almost open at } x\} = \emptyset$ .

A subset  $A$  of  $X$  is called *quasi-open* if  $A \subset \text{Int}\bar{A}$ . For any subset  $A$  of  $X$  define the *quasi-interior*:

$$\mathcal{Q}\text{Int}A = A \cap \text{Int}\bar{A}.$$

For a continuous map  $f : X_1 \rightarrow X_2$  and a subset  $A$  of  $X_1$ , define

$$\mathcal{Q}\text{Int}_fA = (\mathcal{Q}\text{Int}A) \cap f^{-1}(\mathcal{Q}\text{Int}f(A)) = (\mathcal{Q}\text{Int}A) \cap f^{-1}(\text{Int}f(A)).$$

The two definitions agree because  $f^{-1}(f(A)) \supset A$ . In particular:

$$x \in \mathcal{Q}\text{Int}_fA \text{ if and only if } x \in \mathcal{Q}\text{Int}A \text{ and } f(x) \in \text{Int}f(A).$$

PROPOSITION 3.10. The following properties are hold

(1)  $A$  is quasi-open in  $X$  if and only if there exist open subset  $U$  and dense set  $D$  such that  $A = U \cap D$ . In particular, any open set or any dense set is quasi-open.

(2) The arbitrary union of quasi-open sets is quasi-open. If  $A$  is quasi-open and  $U$  is open, then  $A \cap U$  is quasi-open.

(3) The  $\mathcal{Q}\text{Int}A$  is dense in  $\text{Int}\bar{A}$ .  $\mathcal{Q}\text{Int}A$  is the largest quasi-open set contained in  $A$ . In particular,  $\mathcal{Q}\text{Int}(\mathcal{Q}\text{Int}A) = \mathcal{Q}\text{Int}A$ . The set  $A$  is quasi-open if and only if  $A = \mathcal{Q}\text{Int}A$ .  $\mathcal{Q}\text{Int}A = \emptyset$  if and only if  $A$  is nowhere dense.

(4) If  $f : X_1 \rightarrow X_2$  is continuous and  $A \subset X_1$ , then  $\mathcal{Q}\text{Int}_fA$  is a quasi-open subset of  $\mathcal{Q}\text{Int}A$ . If  $A$  is quasi-open, then  $f(\mathcal{Q}\text{Int}_fA) = \mathcal{Q}\text{Int}f(A)$  which is quasi-open in  $X_2$ . If  $U$  is open, then  $\mathcal{Q}\text{Int}_fU$  is open.

*Proof.* (1) Let  $A$  be quasi-open in  $X$ , i.e.,  $A \subset \text{Int}\bar{A}$ . Put  $D = A \cup (X - \bar{A})$ . Then  $D$  is dense in  $X$ . Since  $A$  is quasi-open,  $A =$

$D \cap \text{Int}\bar{A} = (A \cup (X - \bar{A})) \cap \text{Int}\bar{A}$ . Conversely, let  $A = U \cap D$  where  $U$  is open and  $D$  is dense in  $X$ . Then  $A$  is dense in  $U$ , i.e.,  $\bar{A} = \bar{U}$ . Thus  $A \subset U \subset \text{Int}\bar{U} = \text{Int}\bar{A}$ . Since  $A \cap X = A$ , any open set or any dense set is quasi-open.

(2) Let  $\{A_\alpha : \alpha \in \Lambda\}$  be quasi-open, i.e.,  $A_\alpha \subset \text{Int}\bar{A}_\alpha$  for all  $\alpha \in \Lambda$ . Since  $A_\alpha$  is quasi-open for all  $\alpha \in \Lambda$ ,  $\cup_{\alpha \in \Lambda} A_\alpha \subset \cup_{\alpha \in \Lambda} \text{Int}\bar{A}_\alpha \subset \text{Int}(\cup_{\alpha \in \Lambda} \bar{A}_\alpha) \subset \text{Int}(\overline{\cup_{\alpha \in \Lambda} A_\alpha})$ .

Let  $A$  be quasi-open and let  $U$  be open. By (1),  $A = V \cap D$  for some open  $V$  and dense  $D$ . Thus  $A \cap U = (U \cap V) \cap D$  is quasi-open.

(3) Since  $A$  is dense in  $\bar{A}$  and  $\text{Int}\bar{A}$  is open in  $\bar{A}$ ,  $\mathcal{Q}\text{Int}A = A \cap \text{Int}\bar{A}$  is dense in  $\text{Int}\bar{A}$ . We have that  $\mathcal{Q}\text{Int}(\mathcal{Q}\text{Int}A) = \mathcal{Q}\text{Int}A \cap \text{Int}(\overline{\mathcal{Q}\text{Int}A}) = \mathcal{Q}\text{Int}A \cap \text{Int}(\text{Int}\bar{A}) = A \cap \text{Int}\bar{A} \cap \text{Int}\bar{A} = \mathcal{Q}\text{Int}A$ . Clearly,  $A$  is quasi-open if and only if  $A = \mathcal{Q}\text{Int}A$ . Thus  $\mathcal{Q}\text{Int}A$  is quasi-open.

(4) Since  $\mathcal{Q}\text{Int}A$  is quasi-open and  $f^{-1}(\text{Int}f(A))$  is open,  $\mathcal{Q}\text{Int}_f A = \mathcal{Q}\text{Int}A \cap f^{-1}(\mathcal{Q}\text{Int}f(A)) = \mathcal{Q}\text{Int}A \cap f^{-1}(\text{Int}f(A))$  is quasi-open which also shows that  $\mathcal{Q}\text{Int}_f A$  is open when  $A$  is open. By (3),  $f(\mathcal{Q}\text{Int}_f A) = f(\mathcal{Q}\text{Int}A \cap f^{-1}(\text{Int}f(A))) = f(\mathcal{Q}\text{Int}A) \cap \text{Int}f(A) = f(A) \cap \text{Int}f(A) = \mathcal{Q}\text{Int}f(A)$ .  $\square$

Generally,  $\text{Int}_f A \subset \text{Int}A \subset \mathcal{Q}\text{Int}A \subset A$  and  $\text{Int}_f A \subset \mathcal{Q}\text{Int}_f A \subset \mathcal{Q}\text{Int}A \subset A$  for any subset  $A$  of  $X_1$ . The followings are examples in which equalities are not hold by the above inclusion relations.

EXAMPLE 3.11. If  $A = ([0, 1] \cap \mathbb{Q}) \cup [1, 2]$ , then  $\text{Int}A \subsetneq \mathcal{Q}\text{Int}A \subsetneq A$ .

Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = |x|$ . Let  $B = [-1, 1] - \{-\frac{1}{2}, \frac{1}{2}\}$ . Then  $\text{Int}_f B \subsetneq \text{Int}B$ .

Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = |x - 1|$ . Let  $C = ([0, 1] \cap \mathbb{Q}) \cup [1, 2]$ . Then  $\text{Int}_f C \subsetneq \mathcal{Q}\text{Int}_f C$ .

DEFINITION 3.12. Let  $f : X_1 \rightarrow X_2$  be a continuous map. For  $x \in X_1$  and  $V_1 \in \mathcal{U}_{X_1}$ , we call  $f$  is  $V_1$  quasi-open at  $x$  if  $x \in \mathcal{Q}\text{Int}_f V_1(x)$ . If  $f$  is  $V_1$  quasi-open at  $x$  for every  $V_1 \in \mathcal{U}_{X_1}$ , then we call  $f$  is quasi-open at  $x$ . If  $f$  is quasi-open at every  $x \in X_1$ , then we call  $f$  is quasi-open.

THEOREM 3.13. Let  $f : X_1 \rightarrow X_2$  be a continuous map. For  $x \in X_1$  the following conditions are equivalent.

(1)  $f$  is quasi-open at  $x$ .

(2) For all  $A \subset X_1$ ,  $x \in \mathcal{Q}\text{Int}A$  implies  $x \in \mathcal{Q}\text{Int}_f A$ .

(3) If  $U$  is an open neighborhood of  $x$  in  $X_1$ , then  $\mathcal{Q}\text{Int}_f U$  is an open neighborhood of  $x$  in  $X_1$  with  $f(x)$  in  $f(\mathcal{Q}\text{Int}_f U) = \mathcal{Q}\text{Int}f(U)$  quasi-open in  $X_2$ .

(4) If  $U$  is a neighborhood of  $x$  in  $X_1$ , then  $\overline{f(U)}$  is a neighborhood of  $f(x)$  in  $X_2$ .

*Proof.* (1) $\Rightarrow$ (2). Let  $A \subset X_1$  and  $x \in \mathcal{Q}\text{Int}A$ . Assume  $V_1$  is open in  $\mathcal{U}_{X_1}$  with  $V_1(x) \subset \overline{A}$ . Since  $f(x) \in \text{Int}\overline{f(V_1(x))}$ , there exists open  $U_2$  in  $X_2$  such that  $f(x) \in U_2 \subset \text{Int}\overline{f(V_1(x))} \subset \overline{f(V_1(x))} \subset \overline{f(\overline{A})} \subset \overline{f(A)}$ . This means that  $x \in f^{-1}(U_2) \subset f^{-1}(\text{Int}\overline{f(A)})$ , i.e.,  $x \in \mathcal{Q}\text{Int}_f A$ .

(2) $\Rightarrow$ (3). Since  $U$  is open,  $U$  is quasi-open. By Proposition 3.10,  $f(\mathcal{Q}\text{Int}_f U) = \mathcal{Q}\text{Int}f(U)$ . If  $x \in U$ , then  $x \in \mathcal{Q}\text{Int}_f U$ . This means that  $f(x) \in f(\mathcal{Q}\text{Int}_f U) = \mathcal{Q}\text{Int}f(U)$ .

(3) $\Rightarrow$ (4). Let  $U$  be a neighborhood of  $x$ . By (3),  $f(x) \in f(\mathcal{Q}\text{Int}_f U) = \mathcal{Q}\text{Int}f(U)$ . This means that  $f(x) \in \mathcal{Q}\text{Int}f(U) \subset \text{Int}\mathcal{Q}\text{Int}f(U) \subset \overline{f(U)}$ .

(4) $\Rightarrow$ (1). Let  $V_1 \in \mathcal{U}_{X_1}$  be given. Since  $V_1(x)$  is neighborhood of  $x$ ,  $\overline{f(V_1(x))}$  is a neighborhood of  $f(x)$ .  $\square$

**THEOREM 3.14.** Let  $f : X_1 \rightarrow X_2$  be a continuous map. The following conditions are equivalent.

- (1)  $f$  is quasi-open.
- (2) For all  $A \subset X_1$ ,  $\mathcal{Q}\text{Int}A = \mathcal{Q}\text{Int}_f A$ .
- (3) If  $A$  is quasi-open in  $X_1$ , then  $f(A)$  is quasi-open in  $X_2$ .
- (4) If  $U$  is open in  $X_1$ , then  $f(U)$  is quasi-open in  $X_2$ .
- (5) For all  $U$  open in  $X_1$ ,  $U = \mathcal{Q}\text{Int}_f U$ .

*Proof.* (1) $\Rightarrow$ (2). This is obvious from Theorem 3.13 (1)  $\iff$  (2).

(2) $\Rightarrow$ (3). Let  $A$  be quasi-open in  $X_1$ . Since  $A = \mathcal{Q}\text{Int}A = \mathcal{Q}\text{Int}_f A$ ,  $A \subset f^{-1}(\text{Int}\overline{f(A)})$ . This means that  $f(A) \subset \text{Int}\overline{f(A)}$ .

(3) $\Rightarrow$ (4). Since  $U$  is open,  $U$  is quasi-open. By (3),  $f(U)$  is quasi-open.

(4) $\Rightarrow$ (1). Let  $V_1 \in \mathcal{U}_{X_1}$  and  $x \in X_1$ . Since  $V_1(x)$  is an open neighborhood of  $x$ ,  $f(x) \in \text{Int}\overline{f(V_1(x))}$ .

(2) $\Rightarrow$ (5). Since  $U$  open in  $X_1$ ,  $\mathcal{Q}\text{Int}_f U = \mathcal{Q}\text{Int}U = U$ .

(5) $\Rightarrow$ (4). Let  $U$  be open in  $X_1$ . By (5),  $U = \mathcal{Q}\text{Int}_f U \subset f^{-1}(\text{Int}\overline{f(U)})$ . This means that  $f(U) \subset \text{Int}\overline{f(U)}$ , i.e.,  $f(U)$  is quasi-open in  $X_2$ .  $\square$

**THEOREM 3.15.** Let  $f : X_1 \rightarrow X_2$  and  $g : X_2 \rightarrow X_3$  be continuous maps. If both  $f$  and  $g$  are quasi-open, then  $g \circ f$  is quasi-open. If  $g \circ f$  is quasi-open and  $f$  is surjective, then  $g$  is quasi-open.

*Proof.* Let  $f$  and  $g$  be quasi-open and let  $A_1$  be quasi-open in  $X_1$ . By Proposition 3.14 (3),  $f(A_1)$  and  $g(f(A_1))$  are quasi-open in  $X_2$  and  $X_3$ , respectively. Let  $g \circ f$  be quasi-open,  $f$  be surjective and  $U_2$  be an open subset of  $X_2$ . Since  $f$  is continuous,  $f^{-1}(U_2)$  is open in  $X_1$ .

$(g \circ f)(f^{-1}(U_2)) = g(U_2)$  is quasi-open because  $g \circ f$  is quasi-open and  $f$  is surjective.  $\square$

**DEFINITION 3.16.** Let  $f : X_1 \rightarrow X_2$  be a continuous map. We call  $f$  is *almost quasi-open* when for all  $A \subset X_1$ ,  $\text{Int}A \neq \emptyset$  implies  $\text{Int}\overline{f(A)} \neq \emptyset$ .

**THEOREM 3.17.** Let  $f : X_1 \rightarrow X_2$  be a continuous map. The following conditions are equivalent.

- (1)  $f$  is almost quasi-open.
- (2) For every  $V_1 \in \mathcal{U}_{X_1}$ ,  $\{x : f \text{ is } V_1 \text{ quasi-open at } x\}$  is dense in  $X_1$ .
- (3) For every  $V_1 \in \mathcal{U}_{X_1}$ ,  $\text{Int}\{x : f \text{ is } V_1 \text{ quasi-open at } x\}$  is open and dense in  $X_1$ .
- (4) If  $A$  is quasi-open in  $X_1$ , then  $\mathcal{Q}\text{Int}_f A$  is dense in  $A$ .
- (5) For all  $A \subset X_1$ ,  $\mathcal{Q}\text{Int}A \neq \emptyset$  implies  $\mathcal{Q}\text{Int}f(A) \neq \emptyset$ .
- (6)  $B$  is nowhere dense in  $X_2$  implies  $f^{-1}(B)$  is nowhere dense in  $X_1$ .
- (7)  $D$  is open and dense in  $X_2$  implies  $f^{-1}(D)$  is dense in  $X_1$ .

*Proof.* (1) $\Rightarrow$ (2). Let  $V_1 \in \mathcal{U}_{X_1}$  and  $x \in X_1$ . Let  $V = V_1^{-1}$  be open in  $\mathcal{U}_{X_1}$ , with  $V^2 \subset V_1$ . By (1),  $\text{Int}\overline{f(V(x))} \neq \emptyset$ . Since  $\mathcal{Q}\text{Int}_f V(x) = \mathcal{Q}\text{Int}V(x) \cap f^{-1}(\text{Int}\overline{f(V(x))})$  is open and nonempty set, we can find  $x_1 \in \mathcal{Q}\text{Int}_f V(x)$  and notice that  $f(x_1) \in \text{Int}\overline{f(V(x))} \subset \text{Int}\overline{f(V_1(x_1))}$ . Thus,  $f$  is  $V_1$  quasi-open at  $x_1$ . Since  $V$  can be chosen arbitrarily small,  $\{x : f \text{ is } V_1 \text{ quasi-open at } x\}$  is dense.

(2) $\Rightarrow$ (3). Let  $V_1 \in \mathcal{U}_{X_1}$  be given and let  $V = V_1^{-1}$  be open in  $\mathcal{U}_{X_1}$  with  $V^2 \subset V_1$ .  $f$  is  $V$  quasi-open at  $x$  if and only if there is an open neighborhood  $U$  of  $x$  such that  $f(U) \subset \text{Int}\overline{f(V(x))}$ . By (2),  $V(x) \cap \{x : f \text{ is } V_1 \text{ quasi-open at } x\} \neq \emptyset$  for all  $x \in X$ . If  $x_1 \in U \cap V(x)$ , then  $f(U) \subset \text{Int}\overline{f(V_1(x_1))}$ . So  $f$  is  $V_1$  quasi-open at every point of a neighborhood of  $x$ .

(3) $\Rightarrow$ (4). Let  $A$  be quasi-open in  $X_1$ . Since  $\mathcal{Q}\text{Int}_f A \subset A$ ,  $\overline{\mathcal{Q}\text{Int}_f A} \subset \overline{A}$ . For a neighborhood  $U$  of  $x \in A$ , choose  $V_1$  open in  $\mathcal{U}_{X_1}$  such that  $V_1(x) \subset U$ . Since  $A$  is quasi-open in  $X_1$ , there exists  $x_1 \in V_1(x) \cap A$  such that  $f$  is  $V_1$  quasi-open at  $x_1$ . This means that  $f(x_1) \in \text{Int}\overline{f(V_1(x))} \subset \text{Int}\overline{f(U)} = \text{Int}\overline{f(A)}$ . Since  $x_1 \in A = \mathcal{Q}\text{Int}A$ , it follows that  $x_1 \in \mathcal{Q}\text{Int}_f A$ .

(4) $\Rightarrow$ (5). Let  $A$  be a subset of  $X_1$  with  $\mathcal{Q}\text{Int}A \neq \emptyset$ . Put  $A_1 = \mathcal{Q}\text{Int}A$ . By (4),  $\mathcal{Q}\text{Int}_f A_1$  is dense in  $A_1$ . Since  $A_1$  is quasi-open,  $\mathcal{Q}\text{Int}f(A_1) = f(\mathcal{Q}\text{Int}_f A_1) \neq \emptyset$ .

(5) $\Rightarrow$ (1). Let  $A$  be a subset of  $X_1$  with  $\text{Int}A \neq \emptyset$ . Put  $U = \text{Int}A$ . Since  $U$  is nonempty open,  $U$  is quasi-open. By (5),  $\emptyset \neq \mathcal{Q}\text{Int}f(U) \subset \text{Int}\overline{f(A)}$ .

(1) $\Rightarrow$ (6). Let  $\text{Int}\overline{f^{-1}(B)} \neq \emptyset$  and let  $A = f^{-1}(\overline{B})$ . Since  $f$  is continuous,  $A$  is closed in  $X_1$  and  $f^{-1}(B) \subset A$ . Since  $\text{Int}\overline{f^{-1}(B)} \neq \emptyset$ ,  $\text{Int}A \neq \emptyset$ . By (1),  $\emptyset \neq \text{Int}\overline{f(A)} \subset \text{Int}\overline{B}$ .

(6) $\Rightarrow$ (7). Let  $D$  be open and dense in  $X_2$  and put  $B = X_2 - D$ . Then  $B$  is nowhere dense in  $X_2$ . By (6),  $f^{-1}(B)$  is nowhere dense in  $X_1$ . This means that  $\emptyset = \text{Int}\overline{f^{-1}(B)} = \text{Int}f^{-1}(B) = \text{Int}f^{-1}(X_2 - D) = \text{Int}(X_1 - f^{-1}(D))$ .

(7) $\Rightarrow$ (1). Let  $A$  be a subset of  $X_1$  with  $\text{Int}A \neq \emptyset$  and let  $\text{Int}\overline{f(A)} = \emptyset$ . Put  $D = X_2 - \overline{f(A)}$  and  $D$  is open and dense. By (7),  $f^{-1}$  is open and dense. Since  $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$  and  $f^{-1}(D) = X_1 - f^{-1}(\overline{f(A)})$ ,  $f^{-1}(D) \cap A = \emptyset$ . This is a contradiction. Thus  $\text{Int}\overline{f(A)} \neq \emptyset$ .  $\square$

**THEOREM 3.18.** *Let  $f : X_1 \rightarrow X_2$  and  $g : X_2 \rightarrow X_3$  be continuous maps. If both  $f$  and  $g$  are almost quasi-open, then  $g \circ f$  is almost quasi-open. If  $g \circ f$  is almost quasi-open and  $f$  is surjective, then  $g$  is almost quasi-open.*

*Proof.* Let  $f$  and  $g$  be almost quasi-open and let  $Q\text{Int}A$  be a nonempty subset of  $X_1$ . By Theorem 3.17,  $Q\text{Int}f(A)$  and  $Q\text{Int}g(f(A))$  are nonempty subsets of  $X_2$  and  $X_3$ , respectively. Let  $g \circ f$  be almost quasi-open,  $f$  be surjective and let  $U_2 \equiv \text{Int}A_2$  be a nonempty set, where  $A_2 \subset X_2$ . Since  $f$  is continuous and surjective,  $f^{-1}(U_2) = \text{Int}f^{-1}(U_2)$  is nonempty. By Theorem 3.17,  $\emptyset \neq \text{Int}(g \circ f)(f^{-1}(U_2)) = \text{Int}g(U_2) \subset \text{Int}g(A_2)$ .  $\square$

**REMARK 3.19.** If for every dense  $G_\delta$  set  $B$  in  $X_2$   $f^{-1}(B)$  is dense in  $X_1$ , then  $f$  satisfies (7) of Theorem 3.17, so  $f$  is almost quasi-open. The converse is true as well if  $X_1$  is Baire.

**DEFINITION 3.20.** Let  $f : X_1 \rightarrow X_2$  be a continuous map. For  $x \in X_1$  and  $V_1 \in \mathcal{U}_{X_1}$ , we call  $f$  is a  $V_1$  embedding at  $x$  if there exists  $V_2 \in \mathcal{U}_{X_2}$  such that

$$(f^*V_2)(x) = f^{-1}(V_2(f(x))) \subset V_1(x).$$

This just says that the preimage of some neighborhood of  $f(x)$  is contained in  $V_1(x)$ . Clearly  $f$  is a  $V_1$  embedding at  $x$  if and only if the associated surjective map  $f : X_1 \rightarrow f(X_1)$  is a  $V_1$  embedding at  $x$ . If  $f$  is surjective and is a  $V_1$  embedding at  $x$ , then  $f$  is  $V_1$  open at  $x$  because  $V_2(f(x)) \subset f(V_1(x))$ . If  $f$  is a  $V_1$  embedding at  $x$  for all  $V_1 \in \mathcal{U}_{X_1}$ , then we call  $f$  is an embedding at  $x$ .

**THEOREM 3.21.** *Let  $f : X_1 \rightarrow X_2$  be a continuous map. For  $x \in X_1$  the following conditions are equivalent.*



- (1)  $f$  is an embedding at  $x$ .
- (2)  $f^{-1}u[f(x)] = u[x]$ , where  $u[f(x)]$  is an uniform neighborhood of  $f(x)$ .
- (3) For every  $V_1 \in \mathcal{U}_{X_1}$ , there exists  $U_2$  a neighborhood of  $f(x)$  such that  $f^{-1}(U_2) \times f^{-1}(U_2) \subset V_1$ .

If  $f$  is an embedding at  $x$ , then  $f^{-1}(f(x)) = E_f(x)$  is  $\{x\}$ . Conversely if  $f$  is a closed map and  $E_f(x) = \{x\}$ , then  $f$  is an embedding at  $x$ .

*Proof.* If  $V = V^{-1}$  and  $V^2 \subset V_1$ , then  $f^{-1}(U) \subset V(x)$  implies  $f^{-1}(U) \times f^{-1}(U) \subset V_1$ . So (1) $\Rightarrow$ (3). (3) obviously implies (1) which is equivalent to (2), i.e.,  $\{(f^*V_2)(x) = f^{-1}(V_2(f(x))) : V_2 \in \mathcal{U}_{X_2}\}$  is a base for the neighborhood of  $x$ . Clearly if  $f$  is a  $V_1$  embedding at  $x$ , then  $E_f(x) \subset V_1(x)$ . If  $f$  is an embedding at  $x$ ,  $E_f(x) = \{x\}$ . If  $V_1$  is open and  $f$  is closed, then  $f(X_1 - V_1(x))$  is closed in  $X_2$  and  $f(X_1 - V_1(x)) \cap f(x) = \emptyset$  if  $E_f(x) = \{x\}$ . In that case the complement  $U$  satisfies  $f^{-1}(U) \subset V_1(x)$ . The last assertion is already true at the  $V_1$  level as previously discussed.  $\square$

Notice that  $f$  is an embedding at  $x$ , for all  $x \in X_1$  if and only if  $f$  is an embedding, i.e., if and only if the surjective map  $f : X_1 \rightarrow f(X_1)$  is a homeomorphism.

**DEFINITION 3.22.** A continuous map  $f : X_1 \rightarrow X_2$  is called an *almost embedding* if  $U_1$  is open and nonempty in  $X_1$ , then there exists  $U_2$  open in  $X_2$  such that  $f^{-1}(U_2)$  is a nonempty subset of  $U_1$ .

**THEOREM 3.23.** Let  $f : X_1 \rightarrow X_2$  be a continuous map. The following conditions are equivalent.

- (1)  $f$  is an almost embedding.
- (2) For every  $V_1 \in \mathcal{U}_{X_1}$ ,  $\{x \in X_1 : f^{-1}(U) \times f^{-1}(U) \subset V_1 \text{ for some neighborhood } U \text{ of } f(x)\}$  is open and dense in  $X_1$ .
- (3) For every  $V_1 \in \mathcal{U}_{X_1}$ ,  $\{x \in X_1 : f \text{ is a } V_1 \text{ embedding at } x\}$  is dense in  $X_1$ .
- (4) For  $D \subset X_1$ ,  $f(D)$  dense in  $f(X_1)$  implies  $D$  dense in  $X_1$ .
- (5) For all  $U$  open in  $X_1$  the open set:  

$$U^f \equiv X_1 - f^{-1}(\overline{f(X_1 - U)}) = f^{-1}(\text{Int}(X_2 - f(X_1 - U)))$$
is dense in  $U$ .

*Proof.* (1) $\Rightarrow$ (2). If  $U_1$  is open and nonempty, then we shrink to get  $U_1 \times U_1 \subset V_1$ . By (1), there exist  $x \in U_1$  and  $U_2$  a neighborhood of  $f(x)$  such that  $f^{-1}(U_2) \subset U_1$ .

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (4). Assume  $f(D)$  is dense in  $f(X_1)$ . Given  $x_0 \in X_1$  and  $W \in \mathcal{U}_{X_1}$ , we show that  $W(x_0) \cap D \neq \emptyset$ . Choose  $V_1 \in \mathcal{U}_{X_1}$  symmetric such that  $V_1^2 \subset W$ . By (3), there exists  $x_1 \in V(x_0)$  and  $f$  is a  $V_1$  embedding at  $x_1$ . This means that  $f^{-1}(V_2(f(x_1))) = (f^*V_2)(x_1) \subset V_1(x_1)$  for some  $V_2 \in \mathcal{U}_{X_2}$ . Since  $f(D)$  is dense in  $f(X_1)$ , we can find  $y_1 \in f(D) \cap V_2(f(x_1))$ . Thus there exists  $x_2 \in D$  such that  $f(x_2) = y_1$  and  $f(x_2) \in V_2(f(x_1))$ . Therefore  $x \in f^{-1}(V_2(f(x_1))) = (f^*V_2)(x_1) \subset V_1(x_1) \subset V_1(V_1(x_0)) = V_1^2(x_0) \subset W(x_0)$ . It means that  $x_2 \in D \cap W(x_0)$ . Thus  $D$  is dense in  $X_1$ .

(4) $\Rightarrow$ (5). Let  $D = U^f \cap (X_1 - U) = f^{-1}(\text{Int}(X_2 - f(X_1 - U))) \cup (X_1 - U)$ .  $f(D) = [f(X_1) \cap \text{Int}(X_2 - f(X_1 - U))] \cup f(X_1 - U)$  which is dense in  $f(X_1)$ . By (4),  $D$  is dense in  $X_1$ , so  $D \cap U = (U^f \cup (X_1 - U)) \cap U = U^f \cap U$  is dense in  $U$ .

(5) $\Rightarrow$ (1). Let  $U_1$  be open and nonempty in  $X_1$ . Put  $U_2 = \text{Int}(X_2 - f(X_1 - U_1))$ . By (5),  $f^{-1}(U_2)$  is dense in  $U_1$ . Thus  $f^{-1}(U_2)$  is nonempty. □

DEFINITION 3.24. A continuous map  $f$  is called an *almost homeomorphism* if it is a surjective almost embedding.

EXAMPLE 3.25. Let  $X = \{a, b, c\}$  and  $\tau = \{\emptyset, \{a, b, c\}\}$ . Let  $Y = \{1, 2\}$  and  $\sigma = \{\emptyset, \{1, 2\}\}$ . Define a map  $f : (X, \tau) \rightarrow (Y, \sigma)$  as follows:  $f(a) = 1, f(b) = 1$  and  $f(c) = 2$ . Then  $f$  is an almost homeomorphism. However,  $f$  is not a homeomorphism.

THEOREM 3.26. If  $f$  is an almost homeomorphism, then for any  $V_1$  closed in  $\mathcal{U}_{X_1}$

$$\begin{aligned} \{x : f \text{ is } V_1 \text{ open at } x\} &= \{x : f \text{ is a } V_1 \text{ embedding at } x\}, \\ \{x : f \text{ is open at } x\} &= \{x : f \text{ is an embedding at } x\}. \end{aligned}$$

*Proof.* In general, for a surjective continuous map  $f$  and  $V_1 \in \mathcal{U}_{X_1}$ , if  $f$  is  $V_1$  embedding at  $x$ , then there exists  $V_2 \in \mathcal{U}_{X_2}$  such that  $(f^*V_2)(x) = f^{-1}(V_2(f(x))) \subset V_1(x)$ . It means that  $f(x) \in V_2(f(x)) \subset f(V_1(x))$ . Thus  $f$  is  $V_1$  open at  $x$ .

If  $f$  is an almost embedding,  $A$  is closed in  $X_1$ , and  $U$  is open in  $X_2$ , then the following property holds ;

$$\text{if } U \subset f(A), \text{ then } f^{-1}(U) \subset A.$$

If not, then  $U_1 = f^{-1}(U) - A$  is a nonempty open subset of  $X_1$ . Since  $f$  is almost embedding, there exists nonempty open  $U_2$  in  $X_2$  such that  $f^{-1}(U_2) \subset U_1$ . But  $U_2 = f(f^{-1}(U_2)) \subset f(U_1) \subset U \subset f(A)$ . This is a contradiction.

If  $V_1$  is closed and  $f$  is  $V_1$  open at  $x$ , then there exists open  $U$  of  $X_2$  such that  $f(x) \in U \subset f(V_1(x))$ . Hence there exists  $V_2$  open in  $\mathcal{U}_{X_2}$  such

that  $V_2(f(x)) \subset U \subset f(V_1(x))$ . By the above property,  $f^{-1}(V_2(f(x))) \subset V_1(x)$ . Hence  $f$  is  $V_1$  embedding at  $x$ .

$\{x : f \text{ is open at } x\} = \{x : f \text{ is an embedding at } x\}$  is clear. □

**THEOREM 3.27.** *Let  $f : X_1 \rightarrow X_2$  and  $g : X_2 \rightarrow X_3$  be continuous maps. Then*

- (1) *Assume  $f$  is surjective.  $g \circ f$  is an almost homeomorphism if and only if both  $f$  and  $g$  are almost homeomorphisms.*
- (2) *Assume  $g$  is an almost homeomorphism. If  $g \circ f$  is almost open, then  $f$  is almost quasi-open.*
- (3) *Assume  $g$  is an almost homeomorphism. If  $g \circ f$  is almost quasi-open and  $g$  is closed, then  $f$  is almost quasi-open.*

*Proof.* (1) Let  $U_1$  be a nonempty subset of  $X_1$ . Since  $g \circ f$  is an almost homeomorphism, there exists nonempty open  $U_3 \subset X_3$  such that  $(g \circ f)^{-1}(U_3)$  is a nonempty subset of  $U_1$ , i.e.,  $f^{-1}(g^{-1}(U_3)) \subset U_1$ . Put  $U_2 \equiv g^{-1}(U_3)$ . Since  $g$  is continuous,  $U_2$  is a nonempty open subset of  $X_2$ . It follows that  $f^{-1}(U_2) \subset U_1$ . Hence  $f$  is an almost homeomorphism. Let  $U_2$  be a nonempty open subset of  $X_2$ . Since  $f$  is surjective continuous,  $f^{-1}(U_2)$  is nonempty open in  $X_1$ . By the definition of the almost homeomorphism of  $g \circ f$ , there exists nonempty open  $U_3 \subset X_3$  such that  $(g \circ f)^{-1}(U_3) \subset f^{-1}(U_2)$ . It follows that  $g^{-1}(U_3) = f(f^{-1}(g^{-1}(U_3))) \subset f(f^{-1}(U_2)) = U_2$ . Hence  $g$  is an almost homeomorphism.

Conversely, let  $f$  and  $g$  be almost homeomorphisms and let  $U_1$  be a nonempty subset of  $X_1$ . By the definition of the almost homeomorphism of  $f$  and  $g$ , we can find nonempty open  $U_2 \subset X_2$  and  $U_3 \subset X_3$  such that  $f^{-1}(U_2) \subset U_1$  and  $g^{-1}(U_3) \subset U_2$ . This means that  $(g \circ f)^{-1}(U_3) = f^{-1}(g^{-1}(U_3)) \subset f^{-1}(U_2) \subset U_1$ .

(2) Let  $A$  be a subset of  $X_1$  with  $\text{Int}A \neq \emptyset$ . Since  $g \circ f$  is almost open,  $\text{Int}(g \circ f)(\text{Int}A) \neq \emptyset$ . Since  $g$  is surjective and continuous,  $g^{-1}(\text{Int}g(f(\text{Int}A)))$  is nonempty open in  $X_2$ . Since  $g$  is an almost homeomorphism, there exists nonempty open  $U_3$  in  $X_3$  such that  $g^{-1}(U_3) \subset g^{-1}(\text{Int}g(f(\text{Int}A)))$ .

(3) Let  $A$  be a subset of  $X_1$  with  $\text{Int}A \neq \emptyset$ . Put  $U_1 = \text{Int}A$ . Since  $g \circ f$  is almost quasi-open,  $\text{Int}(\overline{g \circ f}(U_1)) \neq \emptyset$ . Since  $g$  is continuous and closed,  $\text{Int}g(f(U_1)) = \text{Int}g(\overline{f(U_1)})$ . Put  $U_3 = \text{Int}g(f(U_1))$ . Then  $U_3$  is nonempty open and satisfies  $U_3 \subset \overline{g(f(U_1))}$  and  $g^{-1}(U_3) \subset f(U_1)$ . This means that  $\emptyset \neq g^{-1}(U_3) \subset \text{Int}f(U_1) \subset \text{Int}f(A)$ . Thus,  $f$  is almost quasi-open. □

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